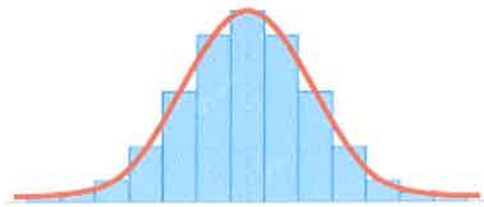
Chapter 5: Continuous Random Variables

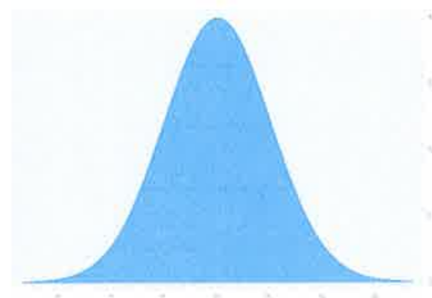
5.1: Continuous Probability Functions

Probability density function (PDF): a probability distribution for a continuous random variable, sometimes thought of as the curve that would be attained from smoothing a histogram obtained through infinite sampling.



Properties of continuous probability distributions

- Notated as $f(x)$, but $f(x) \neq P(X = x)$
- Rather, $P(X = a) = 0$ for all a
- $P(a < X < b)$ is the area under the curve from a to b
- $f(x) \geq 0$ for all values of x
- $f(x) = 0$ for all x not in the sample set
- The total area under the curve equals one.



Key idea: area = probability

<table>
<tr><th>Discrete (pmf)</th><th>Continuous (pdf)</th></tr>
<tr><td>
<table>
<tr><th>X</th><th>P(x)</th></tr>
<tr><td>0</td><td>.01</td></tr>
<tr><td>1</td><td>.04</td></tr>
<tr><td>2</td><td>.15</td></tr>
<tr><td>3</td><td>.8</td></tr>
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$P(X = 3) = f(3) = P(3)$
$P(X \leq 3) = P(X < 3) + P(X = 3)$
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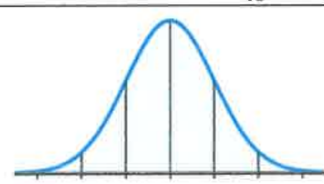
$P(X = 3) = 0$
$P(X \leq 3) = P(X < 3) + P(X = 3)$
$P(X \leq 3) = P(X < 3) + 0$
$P(X \leq 3) = P(X < 3)$
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</table>

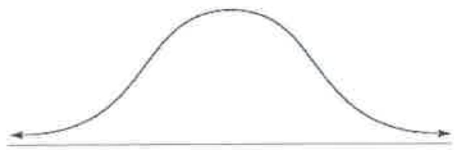
Chapter 6: The Normal Distribution

The normal, a continuous distribution, is the most important of all the distributions. Its graph is bell shaped. You see the bell curve in almost all disciplines. Some of these include psychology, business, economics, the sciences, nursing, and of course, mathematics.

In this chapter, we will study the normal distribution, the standard normal distribution, and applications associated with them.

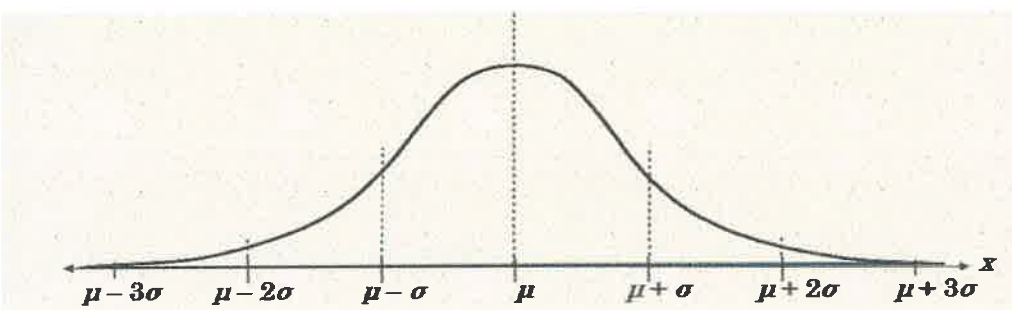
The normal distribution has two parameters (two numerical descriptive measures): the mean (μ) and the standard deviation (σ). If X is a quantity to be measured that has a normal distribution with mean (μ) and standard deviation (σ), we designate this by writing

Normal: $X \sim N(\mu, \sigma)$

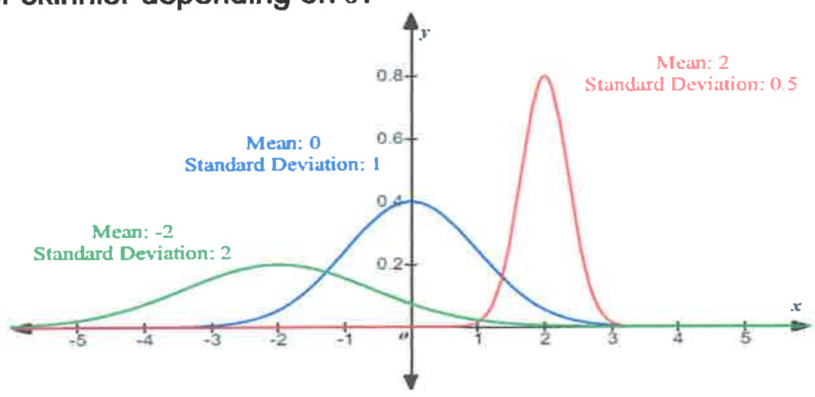


Properties of ANY Normal Distribution:

- The graph for a Normal Distribution is symmetric and bell-shaped.
- The normal curve is symmetric about the mean, μ, such that half of the data is to the left and half falls to the right of the mean, μ.
- The mean, median, and mode are all centered in the middle of a Normal Distribution.
- The total area under the curve is 1.



A change in the standard deviation, σ, causes a change in the shape of the curve; the curve becomes fatter or skinnier depending on σ.

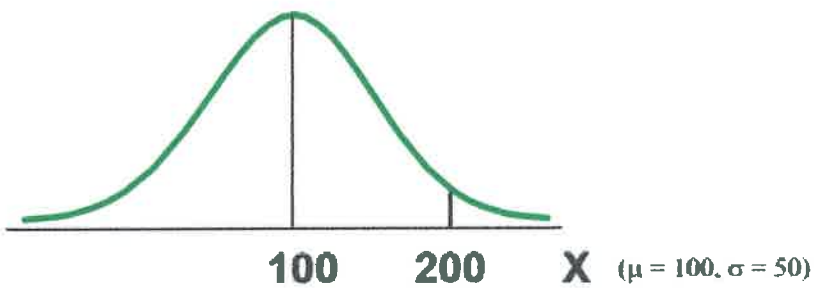


6.1: The Standard Normal Distribution

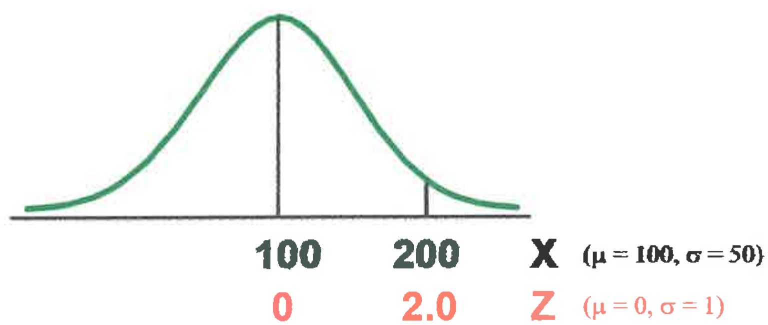
The **Standard Normal Distribution** is a normal distribution of **standardized values** called ***z*-scores**.

A ***z*-score** is measured in units of the standard deviation. The z-score tells you how many standard deviations the value x is above (to the right of) or below (to the left of) the mean μ.

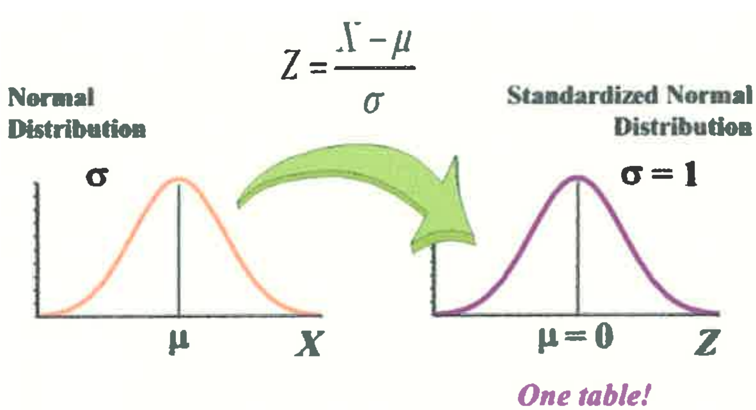
For example, if the mean of a **normal distribution** is 100 and the standard deviation is 50, the value 200 is two standard deviations above (or to the right of) the mean.



The mean, μ, for the standard normal distribution is 0, and the standard deviation, σ, is 1.



The transformation formula:



The transformation $z = \frac{x-\mu}{\sigma}$ produces the distribution $Z \sim N(0,1)$. The value x in the given equation comes from a normal distribution with mean μ and standard deviation σ.

Suppose $X \sim N(5, 2)$. This says that X is a normally distributed variable with mean $\mu = 5$ and standard deviation $\sigma = 2$. Suppose $x = 9$. Then,

$$z = \frac{x - \mu}{\sigma} = \frac{9 - 5}{2} = 2$$

This means that $x = 9$ is **two standard deviations** (2σ) above or to the right of the mean $\mu = 5$.

Now suppose $x = 1$. Let's find the z-score associated with it. $z = \frac{1-5}{2} = \frac{-4}{2} = -2$

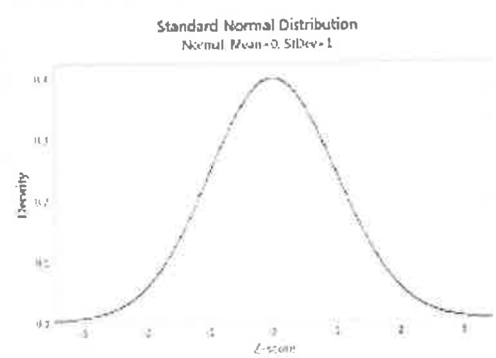
Fill in the blank.

In a normal distribution, x = 5 and z = -1.25. This tells you that x = 5 is 1.25 standard deviations to the left (right or left) of the mean.

In a normal distribution, x = 3 and z = 0.67. This tells you that x = 3 is 0.67 standard deviations to the Right (right or left) of the mean.

Properties of the Standard Normal Distribution (the z-distribution):

- The graph for the Standard Normal Distribution is symmetric and bell-shaped.
- The mean for the Standard Normal Distribution is 0 with a standard deviation of 1.
- The normal curve is symmetric about the mean, μ, such that half of the data is to the left and half falls to the right of the mean, μ.
- The mean, median, and mode are all centered in the middle of the Standard Normal Distribution.
- The total area under the curve is 1.



Example 1: Properties Review

A. The mean of a normal distribution is
 a) 0 b) 0.3 (c) μ d) σ

B. The mean of a standard normal is
 (a) 0 b) 0.3 c) μ d) σ

C. The standard deviation of standard normal distribution is $\sigma = 1$
 a) σ^2 (b) 1 c) μ d) σ $\sigma^2 = 1$

D. The mode of a normal distribution is
 a) 0 b) 0.5 (c) μ d) σ

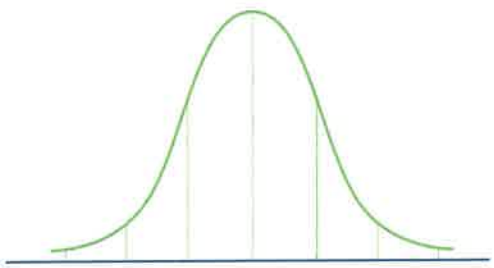
E. The area to the right of μ in a standard normal distribution is
 a) 0 (b) 0.5 c) μ d) σ

True or (False) Another name for the normal distribution is the z-distribution.

6.2: Using the Normal Distribution

Normal Curve - many continuous random variables have relative frequency histograms with a shape like the figure below. They are said to have the shape of a normal curve.

A continuous random variable is **normally distributed**, or has a normal probability distribution, if the relative frequency histogram of the random variable has the shape of a normal curve.

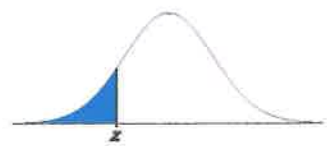


For symmetric distributions with a single peak, such as the normal distribution, the **mean** = **median** = **mode**. Because of this, the mean is the high point of the graph of the distribution.

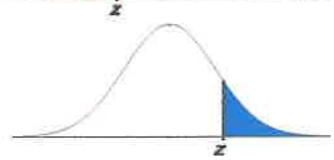
Finding Probabilities Under the Standard Normal Curve When Given z-Scores:

Notation for the Probability of a Standard Normal Random Variable

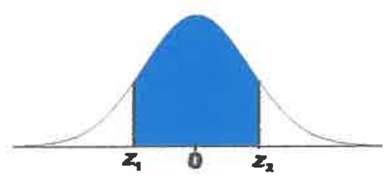
$P(Z < z)$ or $P(Z \leq z)$ represents the probability (area) to the left of z



$P(Z > z)$ or $P(Z \geq z)$ represents the probability (area) to the right of z



$P(z_1 < Z < z_2)$ or $P(z_1 \leq Z \leq z_2)$ represents the probability (area) between z_1 and z_2

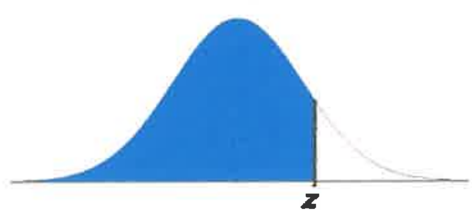


****The z-table (standard normal table) will always give the area to the left of a z-score.**

The **Z – table (the standard normal table)** is a chart used to find the probability (area) under a normal distribution for a continuous random variable.

The area associated with a given z-score refers to the cumulative probability for z.

Cumulative probability is **the area to the left** of a given z-score. $P(Z \leq z)$



z-scores: leftmost column and top row of the z-table.

Areas: the region under the curve; refers to the values in the body of the z-table.

Example 2: Find the following probabilities. Draw and shade the area needed.

A. Area (probability) to the left of the z-score. $P(Z < 0.39)$

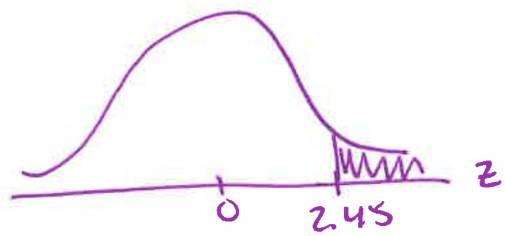
$= 0.6517$

B. Area (probability) to the right of the z-score. $P(Z > 2.45)$

$= 1 - P(Z < 2.45)$

$= 1 - 0.9929$

$= 0.0071$



C. Area (probability) between two z-scores. $P(-0.25 < Z < 0.25)$

$= P(Z < 0.25) - P(Z < -0.25)$

$= 0.5987 - 0.4013$

$= 0.1974$

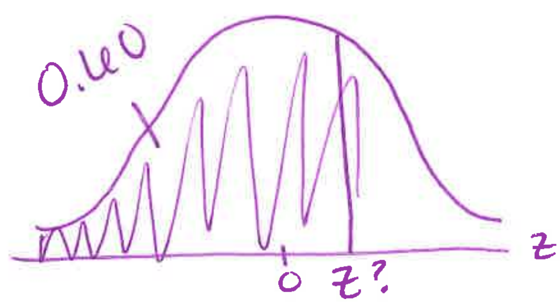
Working backwards from the z-table. Draw and shade the area given.

Example 3: Find the z-score that goes with a particular area.

A. If $P(Z < z_1) = 0.60$, find z_1

look up 0.6000 in middle of z-table

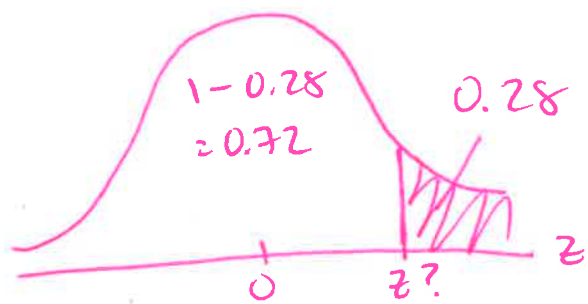
$z = 0.25$



B. If $P(Z > z_1) = 0.28$, find z_1

look up 0.7200 in middle of table

$z = 0.58$



Example 4. Assume that the heights of college women have a normal distribution with mean $\mu = 65$ inches and standard deviation $\sigma = 2.7$ inches.

A. What is the probability that a randomly selected college woman is 62 inches or shorter?

$P(X \leq 62)$

$= P(Z \leq -1.11)$

$= 0.1335$

$z = \frac{x-\mu}{\sigma} = \frac{62-65}{2.7} = -1.11$

B. Find the probability a randomly selected college woman is taller than 70 inches.

$P(X > 70)$

$= P(Z > 1.85)$

$= 1 - P(Z < 1.85)$

$= 1 - 0.9678$

$= 0.0322$

$z = \frac{x-\mu}{\sigma} = \frac{70-65}{2.7} = 1.85$

C. Find the probability that a randomly selected college woman will be between 60 and 68 inches.

$P(60 < X < 68)$

$= P(-1.85 < Z < 1.11)$

$= P(Z < 1.11) - P(Z < -1.85)$

$= 0.8665 - 0.0322$

$= 0.8343$

$z = \frac{x-\mu}{\sigma} = \frac{60-65}{2.7} = -1.85$

$z = \frac{x-\mu}{\sigma} = \frac{68-65}{2.7} = 1.11$

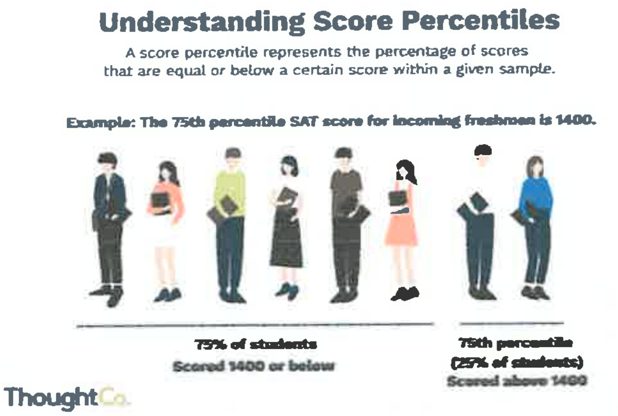
D. Find the probability that a randomly selected college woman will have a height of 64 inches.

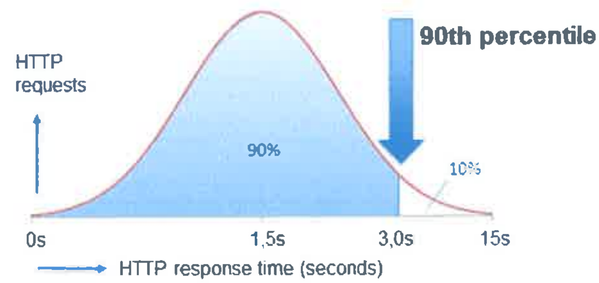
$P(X = 64) = \emptyset$

Reminder: For a continuous random variable $P(X = a) = 0$ for all a.

Percentiles

- Percentile is the percentage of observations that fall **below** a given data point.
- Graphically, percentile is the area below the probability distribution curve to the left of that observation.





Finding Percentiles

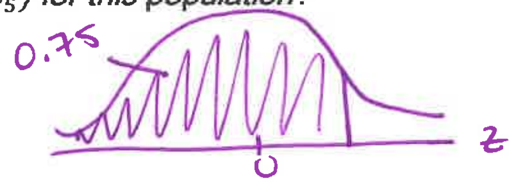
Step 1 Draw a normal curve and shade the area corresponding to the proportion, probability, or percentile given.

Step 2 Use the *z*-table to find the *z*-score that corresponds to the shaded area.

Step 3 Obtain the normal value from the formula $x = \mu + z\sigma$.

Example 5. Suppose that the blood pressure of men aged 18 to 29 years old have a normal distribution with mean $\mu = 120$ and standard deviation $\sigma = 10$.
What **value of blood pressure** is the 75th percentile (P_{75}) for this population?

look up 0.7500 in middle of table



z = 0.67

convert to x

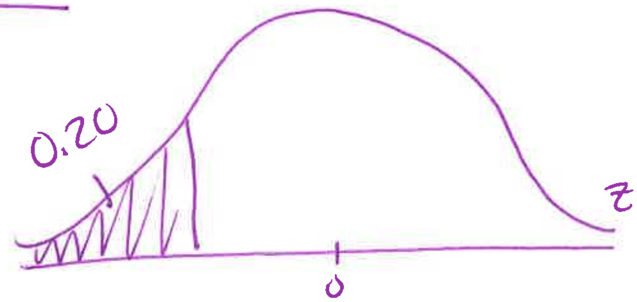
x = 120 + (0.67)(10) = 126.7

Example 6. The heights of a pediatrician's 200 three-year-old females are approximately normally distributed with mean 38.72 inches and standard deviation 3.17 inches. Find the height of a 3-year-old female at the 20th percentile (P_{20}). That is, find the height of a 3-year-old female that separates the bottom 20% from the top 80%.

look up 0.2000 in middle of table

$z = -0.84$

convert to x



$x = \mu + z\sigma$

$= 38.72 + (-0.84)(3.17) = 36.06$ in

Example 7: Suppose a random variable X is approximately normally distributed with $\mu = 11$ and $\sigma = 2$. Find the following probabilities:

(a) $P(X < 8)$

$z = \frac{x-\mu}{\sigma} = \frac{8-11}{2} = -1.50$

$P(z < -1.50)$

$= 0.0668$

(b) $P(10 < X < 12)$

$z = \frac{x-\mu}{\sigma} = \frac{10-11}{2} = -0.50$

$z = \frac{x-\mu}{\sigma} = \frac{12-11}{2} = 0.50$

$P(-0.50 < z < 0.50)$

$= P(z < 0.50) - P(z < -0.50)$

$= 0.6915 - 0.3085$

$= 0.3830$

(c) $P(X > 7.62)$

$z = \frac{x-\mu}{\sigma} = \frac{7.62-11}{2} = -1.69$

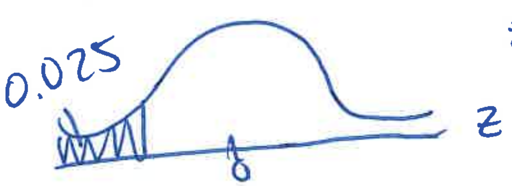
$= P(z > -1.69)$

$= 1 - P(z < -1.69)$

$= 1 - 0.0455$

$= 0.9545$

(d) Find x such that $P(X < C) = .025$



$z = -1.96$

$x = \mu + z\sigma$

$x = 11 + (-1.96)(2)$

$= 7.08$

Example 8: General Electric manufactures a decorative Crystal Clear 60-watt light bulb that it advertises will last 1,500 hours. Suppose that the lifetimes of the light bulbs are approximately normally distributed, with a mean of 1,550 hours and a standard deviation of 57 hours.

$\mu = 1550$ $\sigma = 57$

(a) What probability of the light bulbs will last less than the advertised time?

$P(X < 1500)$
$= P(z < -0.88)$
$= 0.1894$

$z = \frac{x-\mu}{\sigma} = \frac{1500-1550}{57} = -0.88$

(b) What probability of the light bulbs will last more than 1,650 hours?

$P(X > 1650)$
$= P(z > 1.75)$
$= 1 - P(z < 1.75)$
$= 1 - 0.9599$
$= 0.0401$

$z = \frac{x-\mu}{\sigma} = \frac{1650-1550}{57} = 1.75$

(c) What is the probability that a randomly selected GE Crystal Clear 60-watt light bulb will last between 1,625 and 1,725 hours?

$P(1625 < X < 1725)$
$= P(1.32 < z < 3.07)$
$= P(z < 3.07) - P(z < 1.32)$
$= 0.9989 - 0.9066$
$= 0.0923$

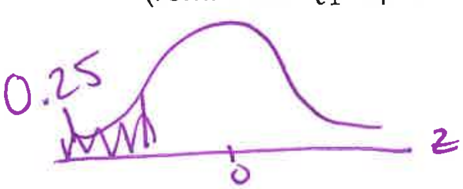
$z = \frac{x-\mu}{\sigma} = \frac{1625-1550}{57} = 1.32$

$z = \frac{x-\mu}{\sigma} = \frac{1725-1550}{57} = 3.07$

(d) What is the probability that a randomly selected GE Crystal Clear 60-watt light bulb will last exactly 1400 hours?

$P(X = 1400) = \emptyset$

(e) What is the length of time a light bulb lasts if it corresponds to the first quartile. (reminder: Q_1 represents the 25th percentile denoted as P_{25})



$z = -0.67$

$x = \mu + z\sigma$
$= 1550 + (-0.67)(57)$
$= 1511.81$