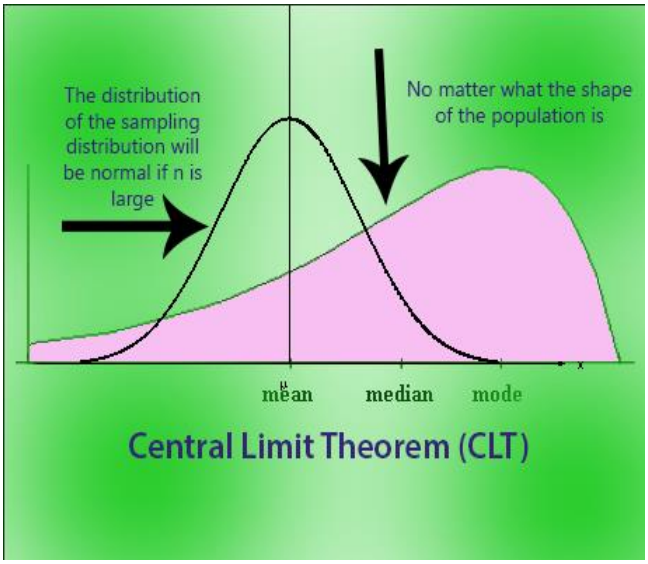
Chapter 7: The Central Limit Theorem

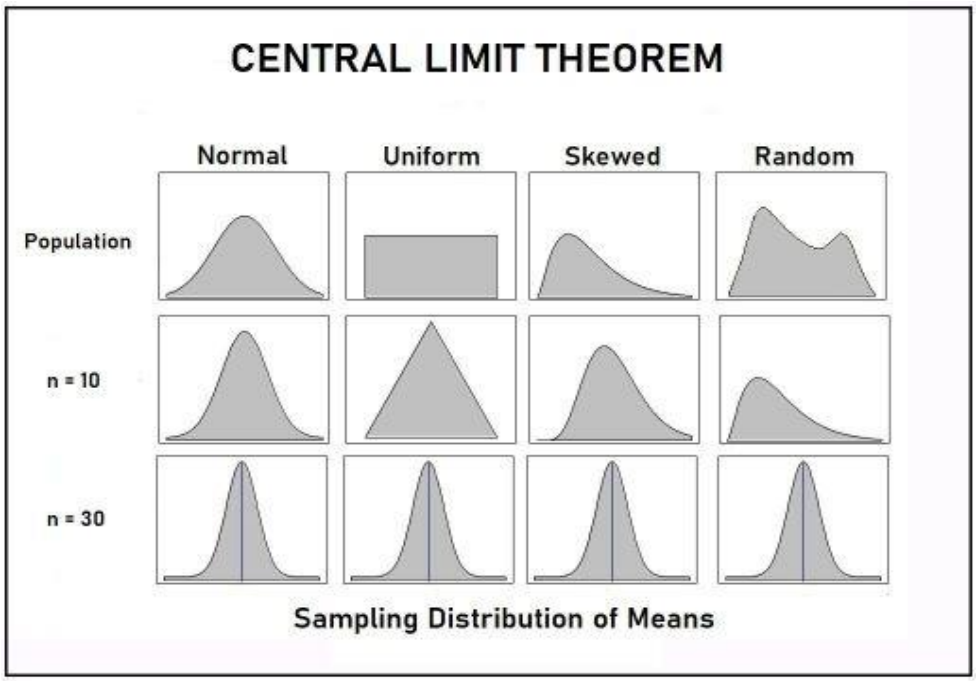
The central limit theorem (CLT for short) is one of the most powerful and useful ideas in all of statistics. There are two alternative forms of the theorem, and both alternatives are concerned with drawing finite samples size *n* from a population with a known mean, *μ*, and a known standard deviation, *σ*. The form we will be working with is stated below:

- If we collect samples of size *n* with a "**large enough *n***," calculate each sample's mean ($\bar{x}\,'s$), and create a histogram of those means, then the resulting histogram will tend to have an approximate normal bell shape.

The size of the sample, *n*, that is required to be "large enough" depends on the original population from which the samples are drawn (the sample size should be at least 30 or the data should come from a normal distribution). If the original population is far from normal, then more observations are needed for the sample means to be normal. Sampling is done with replacement.



It would be difficult to overstate the importance of the central limit theorem in statistical theory. Knowing that data, even if its distribution is not normal, behaves in a predictable way is a powerful tool.



7.1: The Central Limit Theorem for Sample Means (Averages)

Recall:

Population - the complete collection of *all* individuals to be studied.

Sample - a subcollection of members selected from a population.

Parameter - a numerical measurement describing some characteristic of a *population*.

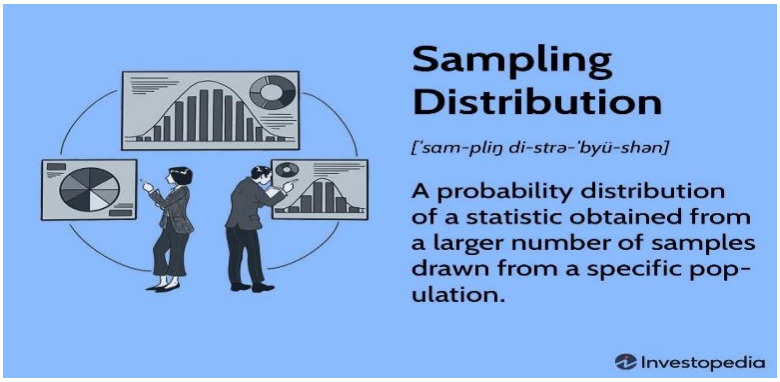
Statistic - a numerical measurement describing some characteristic of a *sample*.

Sampling variability - values of sample statistics vary from sample to sample.

Recall our notation:

	(Population) Parameter	(Sample) Statistic
Proportion	p	$\hat{p}$
Mean	μ	$\bar{x}$
Standard Deviation	σ	s

Sampling Distribution of a statistic - the distribution of all values of the statistic ($\bar{x}$ *or* $\hat{p}$) when all possible samples of the same size *n* are taken from the same population.



Means

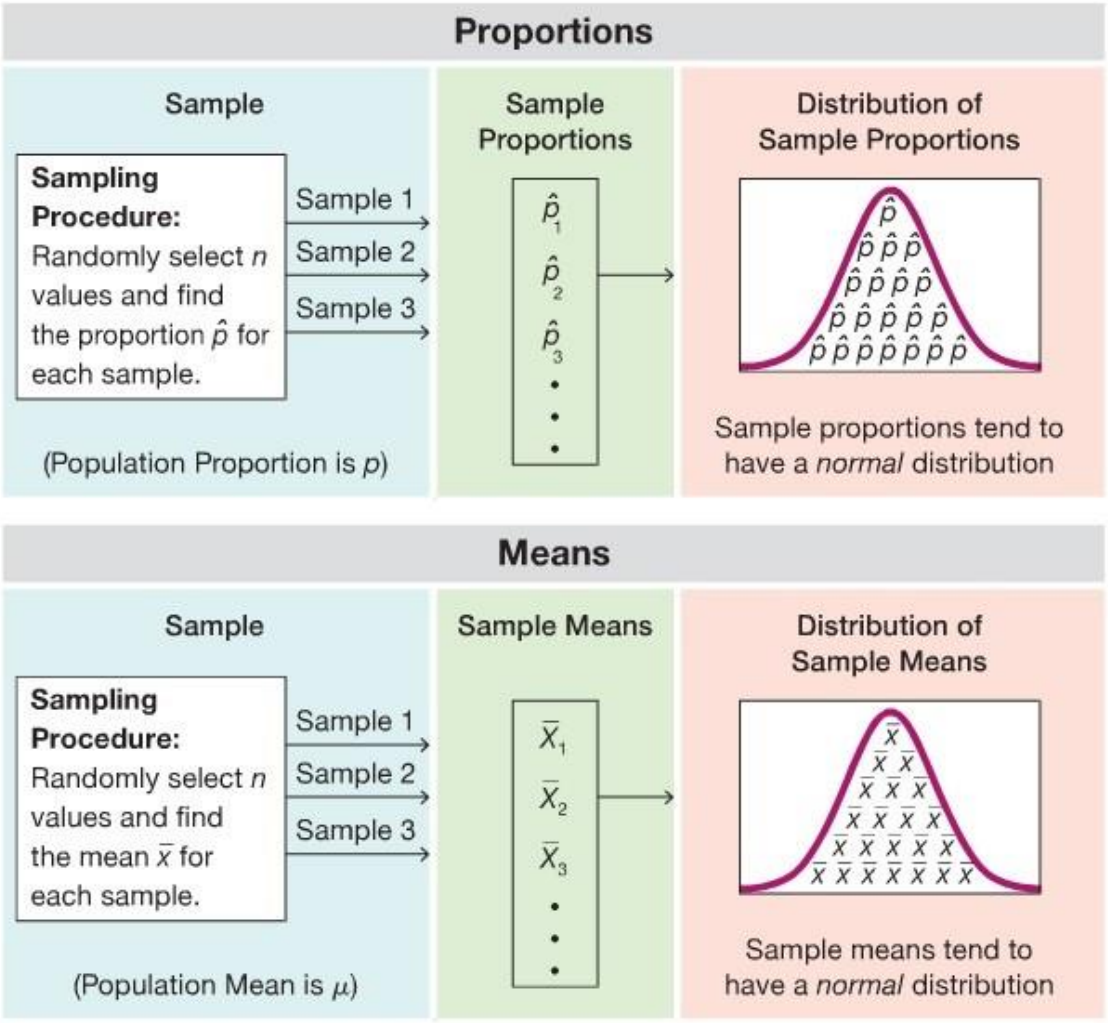
Sampling distribution of the sample mean $\bar{X}$ - the distribution of all possible sample means with all samples having the same sample size *n* taken from the same population.

Proportions

Sampling distribution of the sample proportion $\hat{p}$ - the distribution of all possible sample proportions with all samples having the same sample size *n* taken from the same population.

The parameter of interest is the **population proportion, p**.

In each case, we can compute the statistic $\hat{p}$ (the sample proportion)**:** $\hat{p} = \frac{x}{n}$



Recall the notation from Chapter 6: $\boldsymbol{X \sim N(\mu, \sigma)}$

Suppose X is a random variable with a distribution that may be known or unknown (It can be any distribution). Using a subscript that matches the random variable, suppose:

a. μ_x = the mean of X
b. σ_x = the standard deviation of X

If you draw random samples of size n, then as n increases, the random variable $\boldsymbol{\bar{X}}$ which consists of sample means, tends to be **normally distributed** and

$$\boldsymbol{\bar{X} \sim N\left(\mu_x, \frac{\sigma_x}{\sqrt{n}}\right)}$$

The Central Limit Theorem and the Sampling Distribution of $\bar{x}$

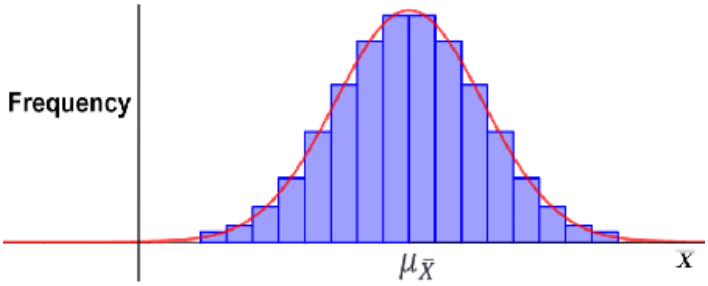
Suppose a simple random sample of size n is to be taken from a large population in which the variable of interest has mean μ and standard deviation σ. Then, if you draw samples of size n, the distribution of the random variable $\bar{X}$, which consists of sample means, is called the **sampling distribution of the mean $\bar{x}$.** The sampling distribution of the mean approaches a normal distribution as n, the sample size, increases.

The sampling distribution of the sample mean $\bar{X}$ will have the following properties:

Shape: (approximately) normal

Center (mean): $\mu_{\bar{x}} = \mu$

Spread (standard deviation): $\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$



Describe the Sampling Distribution of the Sample Mean: $\bar{X} \sim N\left(\mu, \frac{\sigma}{\sqrt{n}}\right)$

The random variable $\bar{X}$ has the following z-score formula: $z_{\bar{x}} = \frac{\bar{x}-\mu_{\bar{x}}}{\sigma_{\bar{x}}} = \frac{\bar{x}-\mu}{\sigma/\sqrt{n}}$

Conditions for Normality:

1. Representative Sample
2. One of the following:
   (a) The population must be normally distributed, OR
   (b) The sample size needs to be *large* enough, $n \geq 30$

The Central Limit Theorem and the Sampling Distribution of $\hat{p}$

Suppose a simple random sample of size *n* is to be taken from a large population in which the true population possessing the attribute of interest is p. Then we can predict three things about the **sampling distribution of the sample proportion $\hat{p}$**:

Shape: (approximately) normal

Center (mean): $\mu_{\hat{p}} = p$

Spread (standard deviation): $\sigma_{\hat{p}} = \sqrt{\frac{p(1-p)}{n}}$

Describe the Sampling Distribution of the Sample Proportion**:** $\boldsymbol{\hat{p} \sim N\left(p,\ \sqrt{\frac{p(1-p)}{n}}\right)}$

The random variable $\hat{p}$ has the following z-score formula: $\boldsymbol{z = \frac{\hat{p}-\mu_{\hat{p}}}{\sigma_{\hat{p}}} = \frac{\hat{p}-p}{\sqrt{\frac{p(1-p)}{n}}}}$

Conditions for Normality

1. Representative sample
2. X, the number of successes, follows a binomial distribution
3. Both np and nq are at least 5

			Sampling Distribution		
Variable	Parameter	Statistic	Center	Spread	Shape
Categorical (example: left-handed or not)	P = population proportion	$\hat{p}$ = sample proportion	p	$\sqrt{\frac{p\,(1-p)}{n}}$	Normal if np ≥ 5 and n(1-p) ≥ 5
Quantitative (example: age)	μ = population mean, σ = population standard deviation	$\bar{x}$ = sample mean	μ	$\frac{\sigma}{\sqrt{n}}$	Normal if n > 30 (Always normal if population is normal)

Notes:

It's IMPORTANT to distinguish clearly between parameters and statistics.

- A parameter is associated with a population. We typically do not know the parameter value in real life, though we may perform calculations assuming a particular parameter value.
- A statistic is associated with a sample. This varies from sample to sample.

The Central Limit Theorem specifies *three* things about the distribution of a sample statistic: shape, center (mean), and spread (standard deviation).

The first step in any CLT problem is to identify which version of the result to use. Determining this involves asking whether the question is about a sample proportion or a sample mean.

- Often the question itself will use the word *proportion* or *mean*, though not always.

7.3: Using the Central Limit Theorem

It is important for you to understand when to use the central limit theorem. If you are being asked to find the probability of the MEAN, use the CLT for the mean. If you are being asked to find the probability of a PROPORTION, use the CLT for proportions.

If you are being asked to find the probability of an INDIVIDUAL VALUE, do not use the CLT. Use the distribution of its random variable.

Sampling for a Long, Long Time: The Law of Large Numbers

The **Law of Large Numbers** says that if you take samples of larger and larger size from any population, then the mean of the sample ($\bar{x}$) tends to get closer and closer to μ. The larger n gets, the smaller the standard deviation gets. As this happens, the sample mean $\bar{x}$ will be close to the population mean μ.

In practice, the Law of Large Numbers says that for any specific population, the larger the sample size, the more you can count on $\bar{x}$ to be an accurate representation of μ.

Example 1. Assume that cans of Dr. Pepper are filled so that the actual amounts have a mean of 12.00 oz and a standard deviation of 1.5 oz. Find the probability that a sample of 36 cans will have a mean amount of at least 12.35 oz.

Example 2. Suppose the population proportion of people who never wear a seatbelt is 30%. In a random sample of 200 drivers, what is the probability that less than 50 individuals say that they never wear a seatbelt when driving.

Example 3. The Centers for Disease Control and Prevention reported that 20.9% of American adults smoked regularly in 2012. Treat this as the parameter value for the current population of American adults. $n = 100$

(a) What symbol represents the population proportion, 0.209?

(b) Describe the sampling distribution of a sample 100 American adults.

(c) Calculate the probability that the sample proportion who smoke will exceed 0.25.

Example 4. A simple random sample of size $n = 49$ is obtained from a population with $\mu = 80$ and $\sigma = 14$.

(a) What is $P(x > 83)$?

(b) What is $P(\bar{x} \leq 75.8)$?

(c) What is $P(78.3 \leq \bar{x} \leq 85.1)$?

Example 5. A blind taste test is done to compare Cola 1 and Cola 2. Among 75 participants surveyed, 60% said they prefer Cola 1. Assuming a 50% chance of a participant choosing Cola 1, what is the probability the proportion observed in a taste test is less than 60%?

Example 6. Based on tests of the Chevrolet Cobalt, engineers have found that the miles per gallon in highway driving are normally distributed, with a mean of 32 miles per gallon and a standard deviation 3.5 miles per gallon.

(a) What is the probability that a randomly selected Cobalt gets more than 34 miles per gallon?

(b) Ten Cobalts are randomly selected and the miles per gallon for each car are recorded. What is the probability that the mean miles per gallon exceeds 34 miles per gallon?

Example 7. A simple random sample of size n=19 is obtained from a population of student heights that is normally distributed with a mean of 65.9 inches and a standard deviation of 4.6 inches. Is the sampling distribution normally distributed? Why?

A. Yes, the sampling distribution is normally distributed because the population is normally distributed.
B. Yes, the sampling distribution is normally distributed because the sample mean is greater than 30.
C. No, the sampling distribution is not normally distributed because the population is not normally distributed.
D. No, the sampling distribution is not normally distributed because the sample size is less than 30.

Example 8. For X with $\mu = 2$, $\sigma = 3$ and a sample size of 25, $P(\overline{X} > 1) =$
Note: round your z score to two decimals places

A. 0.9788.
B. 0.3707.
C. 0.0475
D. 0.6293.
E. none of the above answers are accurate because of a failed assumption.

Example 9. It is known that 64% of Americans personally worry a great deal about federal spending and the budget deficit. Which of the following needs to be satisfied in order to be able to use the Central Limit Theorem to describe the sampling distribution of the proportion of Americans who personally worry a great deal about federal spending and the budget deficit? **Select all the answers that apply.**

A. X follows a binomial distribution
B. $n \geq 30$ or X follows a normal distribution
C. $np \geq 5$ and $nq \geq 5$
D. The sample needs to be representative of all Americans

Example 10. We can "trust" the results of a study using large samples more than we can trust those from a study using smaller samples because of

A. confidence intervals.
B. sampling distributions.
C. the Central Limit Theorem.
D. the Law of Large Numbers.
E. None of the above